Final Exam — Ordinary Differential Equations (WIGDV–07)

Wednesday 2 November 2016, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

Solve the following differential equation for $x > 0$:

$$x^2 y' = y^2 - 6xy + 12x^2.$$

Problem 2 (3 + 4 + 3 points)

Assume that the function $u : [1, \infty) \to \mathbb{R}$ is continuous and satisfies the following inequality:

$$u(x) \leq x^2 + \int_1^x \frac{u(t)}{t} dt \qquad \text{for all} \qquad x \geq 1.$$

We define two new functions:

$$y(x) = \int_1^x \frac{u(t)}{t} dt \qquad \text{and} \qquad \phi(x) = u(x) - y(x).$$

(a) Show that y satisfies the following linear initial value problem:

$$y' - \frac{y}{x} = \frac{\phi(x)}{x}, \qquad y(1) = 0.$$

(b) Compute y in terms of an integral and the function ϕ.

(c) Prove that $u(x) \leq 2x^2 - x$ for all $x \geq 1$.

Problem 3 (6 + 6 + 4 + 4 points)

Let $C([0,1])$ denote the linear space of all continuous functions $y : [0,1] \to \mathbb{R}$. This space becomes a Banach space with the norm

$$\|y\| = \sup \big\{ |y(x)| \ : \ x \in [0,1] \big\}.$$

Consider the integral operator

$$T : C([0,1]) \to C([0,1]), \qquad (Ty)(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) - \int_0^x (x-t)y(t)\, dt.$$

(a) Prove that if $Ty = y$, then y satisfies the following initial value problem:

$$y'' + y = \sin(2x), \qquad y(0) = 1, \qquad y'(0) = 0.$$

(b) Prove that $\|Ty - Tz\| \le \frac{1}{2}\|y - z\|$ for all $y, z \in C([0,1])$. (Hint: $\int_0^x x - t\, dt = \frac{1}{2}x^2$.)

(c) Formulate Banach's fixed point theorem.

(d) Prove that T has a unique fixed point.

Problem 4 (4 + 6 + 10 points)

Consider the following initial value problem:

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t), \qquad \mathbf{y}(\tau) = \boldsymbol{\eta},$$

where A is a constant $n \times n$ matrix.

(a) Explain why e^{At} is a fundamental matrix for the homogeneous equation.

(b) Use variation of constants to prove that the solution is given by

$$\mathbf{y}(t) = e^{A(t-\tau)}\boldsymbol{\eta} + \int_\tau^t e^{A(t-s)}\mathbf{b}(s)\, ds.$$

(c) Compute e^{At} for the 2×2 matrix $A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$.

Problem 5 (12 points)

Consider the following 3rd order equation:

$$u''' + u'' + 8u' - 10u = 6 - 20x - 13e^x.$$

Compute the general solution. If the solution is complex-valued, then also give the solution in real form.

Problem 6 (18 points)

Compute *all* eigenvalues $\lambda \in \mathbb{R}$ and corresponding eigenfunctions u of the following boundary value problem:

$$-x^2u'' - 2xu' = \lambda u, \qquad 1 < x < e, \qquad u(1) = 0, \qquad u(e) = 0.$$

Hint: try solutions of the form $u(x) = x^r$. Treat the cases $\lambda < \frac{1}{4}$, $\lambda = \frac{1}{4}$, and $\lambda > \frac{1}{4}$ separately.

End of test (90 points)

Solution of Problem 1 (10 points)

Method 1: using a y/x substitution. We can rewrite the equation as

$$y' = \left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + 12.$$

Setting $u = y/x$ gives the following differential equation:

$$u' = \frac{y' - u}{x} = \frac{u^2 - 7u + 12}{x}.$$

Separation of variables gives

$$\int \frac{1}{u^2 - 7u + 12} du = \int \frac{1}{x} dx.$$

(3 points)

Partial fraction expansion gives

$$\frac{1}{u^2 - 7u + 12} = \frac{1}{(u-4)(u-3)} = \frac{1}{u-4} - \frac{1}{u-3}.$$

Hence, computing integrals gives:

$$\log|u-4| - \log|u-3| = \log|x| + C \quad \Rightarrow \quad \log\left|\frac{u-4}{u-3}\right| = \log x + C.$$

where we use that $|x| = x$ since we assume that $x > 0$.
(4 points)

Taking exponentials gives

$$\frac{u-4}{u-3} = Kx.$$

where $K = \pm e^C$ or $K = 0$ is another arbitrary constant. Solving for u gives

$$u = \frac{4 - 3Kx}{1 - Kx}.$$

Finally, the solution for y is given by

$$y = xu = \frac{4x - 3Kx^2}{1 - Kx}.$$

(3 points)

Method 2: using the Riccati method. The equation is of Riccati type and it is easy to check that $\phi(x) = 4x$ is a solution.
(1 point)

Now $u = y - \phi$ satisfies the following Bernoulli equation:

$$u' = \frac{u^2}{x^2} + \frac{2u}{x}.$$

(2 points)

Then $z = 1/u$ satisfies the following linear equation:

$$z' + \frac{2}{x}z = -\frac{1}{x^2}.$$

(2 points)

The solution is given by

$$z = \frac{C}{x^2} - \frac{1}{x} = \frac{C - x}{x^2},$$

where C is an arbitrary constant.
(3 points)

Finally, the general solution of y is given by

$$y = u + 4x = \frac{1}{z} + 4x = \frac{x^2}{C - x} + 4x = \frac{4Cx - 3x^2}{C - x}.$$

(2 points)

Remark: the Riccati equation can also be solved using $\phi(x) = 3x$. In fact, this is slightly easier since the equation for u then reads as

$$u' = \frac{u^2}{x^2},$$

which can be solved immediately using separation of variables without reduction to a linear equation.

Solution of Problem 2 (3 + 4 + 3 points)

(a) It trivially follows that $y(1) = 0$. The fundamental theorem of calculus gives

$$y'(x) = \frac{u(x)}{x} = \frac{y(x) + \phi(x)}{x} \quad \Rightarrow \quad y'(x) - \frac{y(x)}{x} = \frac{\phi(x)}{x}.$$

(3 points)

(b) Multiplying the differential equation with $1/x$ gives

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{\phi}{x^2} \quad \Rightarrow \quad \left(\frac{y}{x}\right)' = \frac{\phi}{x^2} \quad \Rightarrow \quad y(x) = x\int_1^x \frac{\phi(t)}{t^2}dt.$$

(4 points)

(c) It is given that $\phi(x) \leq x^2$ for all $x \geq 1$. Therefore, using the monotonicity property of the integral, we get

$$\int_1^x \frac{\phi(t)}{t^2}\,dt \leq \int_1^x \frac{t^2}{t^2}\,dt = \int_1^x \,dt = x - 1,$$

which gives

$$y(x) \leq x\int_1^x \,dt = x^2 - x \quad \text{for all} \quad x \geq 1.$$

(2 points)

This implies that

$$u(x) \leq x^2 + y(x) \leq 2x^2 - x \quad \text{for all} \quad x \geq 1.$$

(1 point)

Solution of Problem 3 (6 + 6 + 4 + 4 points)

(a) Assume that $Ty = y$, or, equivalently,

$$y(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) - x\int_0^x y(t)\,dt + \int_0^x ty(t)\,dt.$$

In particular, setting $x = 0$ gives $y(0) = 1$.
(1 point)

Differentiating once gives

$$y'(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) - \int_0^x y(t)\,dt.$$

(2 points)

In particular, setting $x = 0$ gives $y'(0) = 0$.
(1 point)

Differentiating once more gives

$$y''(x) = \sin(2x) - y(x).$$

(2 points)

(b) Let $y, z \in C([0,1])$ be arbitrary. For all $x \in [0,1]$ we have

$$\begin{aligned}
|(Ty)(x) - (Tz)(x)| &= \left|\int_0^x (x-t)(y(t)-z(t))\,dt\right| \\
&\leq \int_0^x (x-t)|y(t)-z(t)|\,dt \qquad \text{(note: } 0 \leq t \leq x \Rightarrow x - t \geq 0) \\
&\leq \int_0^x (x-t)\|y-z\|\,dt \\
&= \|y-z\|\int_0^x (x-t)\,dt \\
&= \tfrac{1}{2}x^2\|y-z\| \\
&\leq \tfrac{1}{2}\|y-z\|.
\end{aligned}$$

(4 points)

Therefore, taking the supremum over all $x \in [0,1]$ gives

$$\|Ty - Tz\| = \sup_{x\in[0,1]} |(Ty)(x) - (Tz)(x)| \leq \tfrac{1}{2}\|y-z\|.$$

(2 points)

(c) Let D be a closed, nonempty subset in a Banach space B. Let the operator $T : D \to B$ map D into itself, i.e., $T(D) \subset D$, and assume that T is a contraction: there exists a number $0 < q < 1$ such that

$$\|Tx - Ty\| \leq q\|x-y\|, \qquad \forall\, x, y \in D,$$

Then the fixed point equation $Tx = x$ has precisely one solution $\bar{x} \in D$.
(4 points)

Moreover, iterations of T converge to this fixed point:

$$x_0 \in D, \quad x_{n+1} = Tx_n \quad \Rightarrow \quad \lim_{n\to\infty} x_n = \bar{x}.$$

(The last statement is not relevant to this problem.)

(d) We take $D = B = C([0,1])$ and we let $T : B \to B$ be as defined above. Part (b) shows that T is a contraction (we can take $q = \frac{1}{2}$). Therefore, all the assumptions of Banach's fixed point theorem are satisfied. This implies that T has a unique fixed point.
(4 points)

Solution of Problem 4 (4 + 6 + 10 points)

(a) We have that $(e^{At})' = Ae^{At}$, which means that every column of e^{At} satisfies the homogeneous equation $\mathbf{y}' = A\mathbf{y}$.
(2 points)

In addition, e^{At} is invertible (the inverse is given by e^{-At}), which implies that the columns of e^{At} are linearly independent.
(2 points)

(b) We try to find a particular solution of the form $\mathbf{y}_p = e^{At}\mathbf{v}(t)$. On the one hand we have

$$\mathbf{y}_p' = Ae^{At}\mathbf{v} + e^{At}\mathbf{v}' = A\mathbf{y}_p + e^{At}\mathbf{v}'.$$

(1 point)

On the other hand, if $\mathbf{y}_p$ solves the inhomogeneous equation, we have

$$\mathbf{y}_p' = A\mathbf{y}_p + \mathbf{b}(t).$$

Therefore, it follows that

$$e^{At}\mathbf{v}'(t) = \mathbf{b}(t) \quad \Rightarrow \quad \mathbf{v}'(t) = e^{-At}\mathbf{b}(t) \quad \Rightarrow \quad \mathbf{v}(t) = \int_\tau^t e^{-As}\mathbf{b}(s)\,ds$$

(3 points)

The general solution is then given by

$$\mathbf{y} = e^{At}\mathbf{c} + \mathbf{y}_p = e^{At}\mathbf{c} + e^{At}\int_\tau^t e^{-As}\mathbf{b}(s)\,ds = e^{At}\mathbf{c} + \int_\tau^t e^{A(t-s)}\mathbf{b}(s)\,ds,$$

where $\mathbf{c} \in \mathbb{R}^n$ is an arbitrary vector.
(1 points)

Finally, the initial condition $\mathbf{y}(\tau) = \boldsymbol{\eta}$ implies that $\mathbf{c} = e^{-A\tau}\boldsymbol{\eta}$, which completes the proof.
(1 point)

(c) The characteristic polynomial is given by

$$\det(A - \lambda I) = \begin{bmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{bmatrix} = (\lambda+1)^2.$$

Therefore, $\lambda = -1$ is an eigenvalue with multiplicity 2. The generalized eigenspaces of A are given by:

$$A + I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^1 = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

(2 points)

$$(A + I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^2 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

(2 points)

Therefore, the dot diagram is given by

$$\left.\begin{aligned} r_1 &= \dim E_\lambda^1 = 1 \\ r_2 &= \dim E_\lambda^2 - \dim E_\lambda^1 = 1 \end{aligned}\right\} \quad \Rightarrow \quad \begin{matrix} \bullet \\ \bullet \end{matrix}$$

which means that we have one cycle of length 2. In particular, we obtain

$$J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

(2 points)

To construct the matrix Q we start by taking a vector $\mathbf{v} \in E_\lambda^2 \setminus E_\lambda^1$. For example, we can take

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (A+I)\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

Listing these vectors in reverse(!) order gives

$$Q = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}.$$

(2 points)

Finally, since $A = QJQ^{-1}$ we get

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} = e^{-t} \begin{bmatrix} 1-2t & 4t \\ -t & 1+2t \end{bmatrix}.$$

(2 points)

Solution of Problem 5 (12 points)

First we solve the homogeneous equation:

$$u''' + u'' + 8u' - 10u = 0.$$

Using the Ansatz $u(x) = e^{\lambda x}$ we get the following characteristic equation

$$\lambda^3 + \lambda^2 + 8\lambda - 10 = 0.$$

It is easy to guess that $\lambda = 1$ is a root. By means of a long divison we get

$$(\lambda - 1)(\lambda^2 + 2\lambda + 10) = 0 \quad \Leftrightarrow \quad (\lambda - 1)((\lambda + 1)^2 + 9) = 0.$$

Hence, the roots are $\lambda = 1$ and $\lambda = -1 \pm 3i$. Therefore, the homogeneous equation has the following solution:

$$u_h(x) = c_1 e^x + c_2 e^{(-1+3i)x} + c^2 e^{(-1-3i)x}.$$

(4 points for correct u_h)

As a particular solution we try the following:

$$\begin{aligned} u_p(x) &= Ax + B + Cxe^x, \\ u_p'(x) &= A + C(x+1)e^x, \\ u_p''(x) &= C(x+2)e^x, \\ u_p'''(x) &= C(x+3)e^x. \end{aligned}$$

(2 points for a correct Ansatz)

Substitution into the equation gives

$$-10B + 8A - 10Ax + 13Ce^x = 6 - 20x - 13e^x.$$

Comparing like terms on both sides gives $A = 2$, $B = 1$, and $C = -1$.
(4 points for correct coefficients)

Finally, the general solution in real form is given by

$$u(x) = u_h(x) + u_p(x) = c_1 e^x + c_2 e^{-x}\cos(3x) + c_3 e^{-x}\sin(3x) + 1 + 2x - xe^x.$$

(2 points)

Solution of Problem 6 (18 points)

Using the Ansatz $u(x) = x^r$ gives the following characteristic equation:

$$r^2 + r + \lambda = 0 \quad \Rightarrow \quad (r + \tfrac{1}{2})^2 + \lambda - \tfrac{1}{4} = 0 \quad \Rightarrow \quad r_{1,2} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \lambda}.$$

(2 points)

We now have to consider three different cases:

Case 1: $\lambda < \frac{1}{4}$. In this case the roots are real and distinct. The general solution of the differential equation is given by

$$u(x) = c_1 x^{r_1} + c_2 x^{r_2}.$$

The boundary conditions give

$$\begin{bmatrix} 1 & 1 \\ e^{r_1} & e^{r_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $r_1 \neq r_2$ we have that the determinant of the coefficient matrix is nonzero. This implies that $c_1 = c_2 = 0$, and thus $u(x) \equiv 0$. Since we only obtain trivial solutions we conclude that $\lambda < \frac{1}{4}$ is *not* an eigenvalue!
(4 points)

Case 2: $\lambda = \frac{1}{4}$. In this case we have $r_1 = r_2 = -\frac{1}{2}$ and we only find one solution, namely $u(x) = x^{-1/2}$. Then it follows from the theory of Euler equations that $v(x) = x^{-1/2} \log x$ is a second solution. (Alternatively, this can be checked by reduction of order; see below.)
(3 points)

Hence, the general solution is

$$u(x) = c_1 x^{-1/2} + c_2 x^{-1/2} \log x.$$

The boundary conditions give

$$\begin{bmatrix} 1 & 0 \\ e^{-1/2} & e^{-1/2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the coefficient matrix has a nonzero determinant, it follows that $c_1 = c_2 = 0$ and thus $u(x) \equiv 0$. Since we only obtain trivial solutions we conclude that $\lambda = \frac{1}{4}$ is *not* an eigenvalue!
(3 points)

We apply reduction of order to find a second solution:

$$\begin{aligned} v(x) &= c(x)x^{-1/2} \\ v'(x) &= c'(x)x^{-1/2} - \tfrac{1}{2}c(x)x^{-3/2} \\ v''(x) &= c''(x)x^{-1/2} - c'(x)x^{-3/2} + \tfrac{3}{4}c(x)x^{-5/2} \end{aligned}$$

Therefore,

$$-x^2 v'' - 2xv' = \tfrac{1}{4}v \quad \Rightarrow \quad c''(x)x + c'(x) = 0.$$

As a solution we can take $c(x) = \log x$ so that $v(x) = x^{-1/2} \log x$.

Case 3: $\lambda > \frac{1}{4}$. In this case the roots form a complex conjugate pair:

$$r_{1,2} = -\tfrac{1}{2} \pm \omega i \quad \text{where} \quad \omega = \sqrt{\lambda - \tfrac{1}{4}} > 0.$$

The general solution, in real-valued form, is therefore given by

$$u(x) = c_1 x^{-1/2} \cos(\omega \log x) + c_2 x^{-1/2} \sin(\omega \log x).$$

(3 points)

The boundary conditions give

$$\begin{bmatrix} 1 & 0 \\ e^{-1/2}\cos(\omega) & e^{-1/2}\sin(\omega) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Nontrivial solutions exist if and only if $\omega = n\pi$, where $n = 1, 2, 3, \ldots$. (Recall that $\omega > 0$!) In this case it follows that $c_1 = 0$ and we just take $c_2 = 1$. In conclusion we get the eigenvalues

$$\lambda_n = \tfrac{1}{4} + n^2\pi^2, \qquad n = 1, 2, 3, \ldots$$

and the corresponding eigenfunctions are

$$u_n(x) = x^{-1/2} \sin(n\pi \log x).$$

(3 points)